Following some algebra, Eq. (2) takes the form

$$\frac{\delta\langle W\rangle}{\delta l}=\frac{\pi}{2\langle\mu\rangle}\{k_3^2+2k_3k_3^*+\langle k_3^*k_3^*\rangle\}=G. \quad (9)$$

The expression within braces in Eqs. (8) and (9) differs from the complete square of k'_3 by the presence of a correlation between fluctuations of the elastic constants at different points of the body.

Thus, the Griffith–Irwin crack criterion is described by Eq. (9) for a longitudinal crack in a randomly heterogeneous body, i.e., the crack begins to grow at the point when the function of the local characteristics occurring within the braces reaches a value G.

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THERMOMECHANICAL BEHAVIOR OF A RECTANGULAR VISCOELASTIC PRISM EXPOSED TO REPEATED STRETCHING AND CONTRACTION

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Vibrational heat production is an important problem in studying the efficiency of viscoelastic elements of structures experiencing cyclic loads. The design of heat regimes constitutes one of the fundamental problems in the construction of such types of vibrational-proof systems as laminated rods, plates, and shells [1] and fiberglass and rubber–metal products, in particular, shock absorbers [2, 3]. Calculation of the critical parameters beyond which a rapid growth in temperature occurs (the phenomenon of thermal explosion), which leads to partial or complete loss of the supporting power of the product as a result of softening of the material, is of particular interest. A variational method has been used [4] to calculate heat production in a two-dimensional shock absorber. The boundary conditions are satisfied on the basis of the St. Venant principle. In the current work, the stress–strain state, self-heating temperature field, and thermal instability of a long rectangular prism being periodically loaded (plane deformation) are investigated.

§ 1. The fundamental thermoelastic equations are presented in [5]. We may obtain the fundamental thermoviscoelastic equations when ν =const by replacing the shear modulus μ by an operator μ^*. We will find the solution of these equations for a plate $|\xi|\le 2L$, $|\eta|\le 2H$ under the boundary conditions

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$$u_x=0, \quad u_y=\pm La(t) \text{ when } \quad y=\pm y_0; \qquad (1.1)$$
$$\sigma_x=0, \quad \sigma_{xy}=0 \text{ when } \quad x=\pm 1,$$

where $x=\xi/L$ and $y=\eta/L$ are dimensionless coordinates and $y_0=H/L$.

The solution of the translational equilibrium equations, with sufficient functional arbitrariness in order to satisfy the boundary conditions (1.1), has the form

$$\frac{u_x}{L}=\sum_{n=1}^{\infty}\left[B_n\left(\frac{3-4\nu}{k_n}\operatorname{sh}k_nx-x\operatorname{ch}k_nx\right)-D_n\operatorname{sh}k_nx\right]\cos k_ny+\sum_{j=1}^{\infty}(C_jy\operatorname{sh}\lambda_jy+A_j\operatorname{ch}\lambda_jy)\sin\lambda_jx; \qquad (1.2)$$

$$\frac{u_y}{L}=2\gamma_0(1-2\nu)y+\sum_{n=1}^{\infty}(B_nx\operatorname{sh}k_nx+D_n\operatorname{ch}k_nx)\sin k_ny+\sum_{j=1}^{\infty}\left[C_j\left(\frac{3-4\nu}{\lambda_i}\operatorname{sh}\lambda_jy-y\operatorname{ch}\lambda_jy\right)-A_j\operatorname{sh}\lambda_jy\right]\cos\lambda_jx,$$

where γ_0, A_j, C_j, B_n, and D_n are unknown coefficients; $k_n=(2n-1)\pi/2y_0$, and $\lambda_j=\pi_j$. We obtain the equations for the stresses after substituting the motions (1.2) in the equations of state. We satisfy the boundary conditions, as was done in [6], arriving at an infinite system of algebraic equations

$$t_nx_n=\sum_{j=1}^{\infty}y_j\left[\frac{k_n^2}{(k_n^2+\lambda_j^2)^2}-\frac{1-\nu}{k_n^2+\lambda_j^2}\right]+\frac{4\nu a(t)}{y_0k_n^2}; \qquad (1.3)$$
$$s_jy_j=\sum_{n=1}^{\infty}x_n\left[\frac{k_n^2}{(k_n^2+\lambda_j^2)^2}-\frac{1-\nu}{k_n^2+\lambda_j^2}\right],$$

where

$$t_n=-\frac{1}{4k_n}\left(\frac{k_n}{\operatorname{sh}^2k_n}+\operatorname{cth}k_n\right);$$
$$s_j=\frac{y_0}{4}\left[\frac{3-4\nu}{\lambda_j}\operatorname{th}\lambda_jy_0-\frac{y_0}{\operatorname{ch}^2\lambda_jy_0}\right].$$

It has been proved [7, 8] that systems of the form (1.3) are entirely regular and that the principal parts of the asymptotic expansions for x_n and y_j (for large n and j) can be represented in the form

$$x_n=\frac{b_0}{k_n^\alpha};\; y_j=\frac{d_0}{\lambda_j^\alpha}, \qquad (1.4)$$

where b_0, d_0, and α are constants, while α is a positive root of the transcendental equation

$$(3-4\nu)\cos\frac{\pi\alpha}{2}=\alpha^2-(1-2\nu)^2.$$

We note that the following inequality holds for α when $0\le\nu\le 0.5$:

$$0.5<\alpha\leqslant 1. \qquad (1.5)$$

If we establish the asymptotic behavior of the unknowns x_n and y_j and make the following substitution in Eqs. (1.3):

$$x_n=\frac{x_pk_p^\alpha}{k_n^\alpha}(n>p),\; y_j=\frac{y_q\lambda_q^\alpha}{\lambda_j^\alpha}$$

we will be able to noticeably improve the method of simple reduction. In contrast to the method of simple reduction, the improved method of reduction will allow us to determine the values of all the unknowns x_n and y_j by solving a system of p+q equations. The solution of the finite system will be significantly facilitated if we are able to use the method of successive approximations. Modern computers allow us to easily solve a system of several hundred equations by means of this method.

A complete analysis of the stress–strain state at any point of a body, including a corner, can be completely analyzed on the basis of the asymptotic equations (1.4). A detailed analysis of the behavior of the stresses as a corner is approached, will be carried out only for the components σ_y, since the behavior of the other components is analogously investigated. We have

$$\sigma_y^*=J^*\frac{\sigma_y}{2\mu_0}=2(1-\nu)\gamma_0-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^nx_n$$

$$\times\left[xk_n\frac{\operatorname{sh}k_nx}{\operatorname{sh}k_n}-k_n\left(\operatorname{cth}k_n-\frac{1}{k_n}\right)\frac{\operatorname{ch}k_nx}{\operatorname{sh}k_n}\right]\cos k_ny+\frac{y_0}{4}\sum_{j=1}^{\infty}(-1)^jy_j$$

$$\times\left\{[\lambda_jy_0\operatorname{th}\lambda_jy_0+2(1-\nu)]\frac{\operatorname{ch}\lambda_jy}{\operatorname{ch}\lambda_jy_0}-\lambda_jy\frac{\operatorname{sh}\lambda_jy}{\operatorname{ch}\lambda_jy_0}\right\}\cos\lambda_jx. \tag{1.6}$$

Here J is an operator inverse to μ^*, μ_0 is the instantaneous shear modulus, and

$$\gamma_0=\frac{a(t)}{2(1-2\nu)y_0-\frac{\nu}{2}\sum_{n=1}^{\infty}x_n/k_n^2}.$$

The series in Eq. (1.6) rapidly converge within the region $|x|<1$, $|y|<y_0$ so that we obtain, as in the case $y=y_0$,

$$\sigma_y^*\approx\frac{\gamma_0y_0(1-\nu)d_0}{2}\sum_{j=1}^{\infty}(-1)^j\frac{\cos\lambda_jx}{\lambda_j^{\alpha}}. \tag{1.7}$$

Bearing in mind inequality (1.5), we see that reliable numerical values of the sums (1.7) can be obtained only after we have improved the convergence of these series [9]. The use of well-known expansions [10] results in a computational variant of Eq. (1.6) when $y=y_0$:

$$\sigma_y^*=2(1-\nu)\gamma_0+\frac{(1-\nu)y_0\gamma_0}{2}\sum_{j=1}^{q}(-1)^j\left(y_j-\frac{y_q\lambda_q^{\alpha}}{\lambda_j^{\alpha}}\right)\cos\lambda_jx+y_q\lambda_q^{\alpha}F(x),$$

where

$$F(x)=\frac{(1-x)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha)\cos\frac{\pi\alpha}{2}}-\frac{\sqrt{\pi}}{2^{\alpha+1}\Gamma\left(\alpha+\frac{1}{2}\right)\cos\frac{\pi\alpha}{2}}+\sum_{j=1}^{N}\left[\frac{(-1)^j}{\lambda_j^{\alpha}}-\sqrt{\frac{\pi\lambda_j}{2}}\frac{J_{\alpha-\frac{1}{2}}(\lambda_j)}{\lambda_i^{\alpha}\cos\frac{\pi\alpha}{2}}\right]\cos\lambda_jx. \tag{1.8}$$

The inequality

$$\left|\sum_{j=q,N}^{\infty}(-1)^j\frac{\cos\lambda_jx}{\lambda_j^{\alpha+1}}\right|<\varepsilon,$$

where ε is the specified computational precision of the stresses, will serve as the criterion for selecting q and N. In this case the boundary conditions (1.1) are satisfied with the same accuracy. As $x\to 1$, Eqs. (1.8) imply that

$$\sigma_y^*\to\frac{(1-\nu)\gamma_0y_0d_0}{4\Gamma(\alpha)\cos\frac{\pi\alpha}{2}}(1-x)^{\alpha-1}. \tag{1.9}$$

The remaining stress tensor components behave in a similar way as a corner is approached. Thus, stresses in the plate near the point separating the boundary conditions have the same singularity as in the problem for a quarter-plane [11]. The resulting solution is suitable for the analysis of plane-stress and plane-strain states, with our notation.

§ 2. The dissipation function and energy equation are analogously determined [6] in the case of cyclic loading $a(t)=a_0\cos\omega t$. If we average the equations with respect to a cycle and adjoin to it the initial and boundary conditions, we arrive at

$$\frac{\partial\theta}{\partial t}=\varkappa\Delta\theta+\frac{1}{c\rho}D_1, \tag{2.1}$$

$$\begin{aligned}&\frac{\partial\theta}{\partial\xi}\pm h_1(\theta-\theta_0)=0\ \text{when}\ \xi=\pm L;\\&\frac{\partial\theta}{\partial\eta}\pm h_2(\theta-\theta_0)=0\ \text{when}\ \eta=\pm H,\end{aligned} \tag{2.2}$$

where

$$D_1=\frac{\omega\mu_2A\gamma}{4\mu_1^2}\left(\sigma_x^{02}+\sigma_y^{02}+\sigma_z^{02}+2\sigma_{xy}^{02}-\frac{\nu}{1+\nu}\sigma^{02}\right);$$

$\varkappa = k/c\rho$ is thermal diffusivity, c is heat capacity, ρ is density, A is the thermal equivalent of mechanical work, γ is that part of the scattering energy that becomes thermal energy [12], h_i is the heat-transfer coefficient, and σ_i^0 are the stresses, equal to the elastic stresses when $\mu = \mu_1$.

Equation (1.9) demonstrates that the dissipation function has a singularity at a corner of the form

$$D_1 \to D_0[(1-x)^2+(y_0-y)^2]^{1-\alpha} \quad \text{when} \quad x \to 1, \ y \to y_0. \tag{2.3}$$

We may prove that temperature is finite when $h_1^2+h_2^2 \neq 0$ at all points of the region by using methods from potential theory and taking into account Eq. (1.5). Well-known numerical methods may be used to solved the problem (2.1), (2.2). It is necessary to replace values of the function D_1 by its mean integral value calculated using Eq. (2.3) within a sufficiently small neighborhood of the corner. This "smoothing" of the source function will not lead to significant error in most practical calculations. Such a solution may be obtained by the solution superposition method [9].

The level of heating will be such that the temperature dependence of the physicomechanical properties can be disregarded within a sufficiently wide range of load parameters. Here the self-heating temperature field is determined by solving the linear system (2.1)-(2.2).

§ 3. Disregarding in the energy equation the dependence of the properties of the material on temperature will not give a correct quantitative and qualitative description of vibrational heating in the case of intensive loading. We thus assume that

$$\mu_2 = \mu_{02}\Phi[\beta(\theta - \theta_0)] \tag{3.1}$$

for the solutions (2.1)-(2.2), arriving at the problem

$$\begin{gathered} \Delta u + \lambda f(x, y)\Phi(u) = 0; \\ \frac{\partial u}{\partial x} \pm B_1 u = 0 \quad \text{when} \quad x = \pm 1; \\ \frac{\partial u}{\partial y} \pm B_2 u = 0 \quad \text{when} \quad y = \pm y_0, \end{gathered} \tag{3.2}$$

where Φ is a given function, β and μ_{02} are experimental constants, and

$$\begin{gathered} x = \frac{\xi}{L}, \ y = \frac{\eta}{L}, \quad u = \beta(\theta - \theta_0), \quad B_1 = Lh_1, \quad B_2 = Lh_2, \\ \lambda = \frac{\omega\mu_{02}a_0^2L^2A\beta\gamma}{k}, \quad f(x, y) = \frac{1}{4\mu_1^2a_0^2}\left(\sigma_x^{02} + \sigma_y^{02} + \sigma_z^{02} + 2\sigma_{xy}^{02} - \frac{\nu}{1+\nu}\sigma^{02}\right) \end{gathered} \tag{3.3}$$

are dimensionless variables. The boundary conditions (3.3) will be henceforth written in the more general form

$$\frac{\partial u}{\partial n} + Bh(x, y)u = 0 \quad \text{on} \quad S. \tag{3.4}$$

The problem (3.2), (3.4) is also encountered in studying the temperature fields of viscous liquids, gases, and gas mixtures [13], in the electrical heating of conductors and electrolytes [14], in calculating internal heating of icebergs [15], and other fields.

Critical values λ_* of the parameter λ above which no positive solutions of the nonlinear system (3.2), (3.4) exist are of greatest interest. Critical values in a number of one-dimensional problems for actual functions Φ have been calculated [1, 2, 16, 17]. A survey of the mathematical aspects of this problem has also been given [18]. In most studies it has been assumed that a trivial solution u_0 is known, which is equivalent to $\Phi(u_0) = 0$. This condition is not satisfied for heat sources operating in viscoelastic media.

We may prove that if the function $u/\Phi(u)$ is bounded,

$$\lambda_* < \chi_0, \tag{3.5}$$

where χ_0 is the minimal eigenvalue of the associated linear problem

$$\begin{gathered} \Delta v + \chi f(x, y)v = 0 \quad \text{in} \quad V; \\ \frac{\partial v}{\partial n} + Bh(x, y)v = 0 \quad \text{on} \quad S. \end{gathered} \tag{3.6}$$

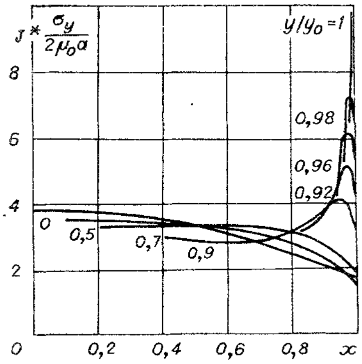

Fig. 1

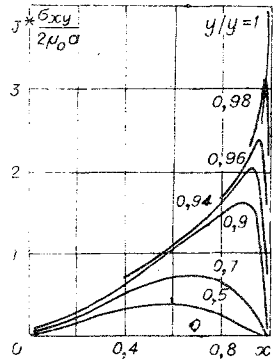

Fig. 2

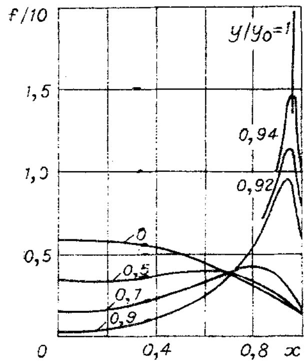

Fig. 3

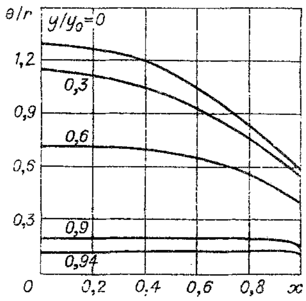

Fig. 4

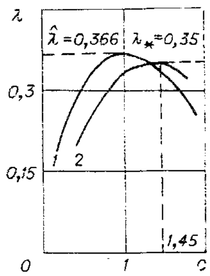

Fig. 5

Under these constraints an equation has been obtained [14] for the upper bounds $\hat{\lambda}$ of λ_*,

$$\lambda_* \leqslant \hat{\lambda} = \chi_0 \max_{u \geqslant 0} \frac{u}{\Phi(u)}. \tag{3.7}$$

The Inequality (3.7) provides an extremely good approximation for λ_*, particularly at low B. Nonetheless, the method of obtaining this equation does not allow us to construct approximations of higher orders.

We will write Eq. (3.1) in the form

$$\Delta u + \chi f u = f[\chi u - \lambda\Phi(u)]$$

and solve the initial system by the method of successive approximations. As a result, we arrive at the linear boundary-value problem

$$\Delta u_0 + \chi f u_0 = 0 \quad \text{in} \quad V; \tag{3.8}$$

$$\frac{\partial u_0}{\partial n} + Bhu_0 = 0 \quad \text{on} \quad S;$$

$$\Delta u_n+\chi f u_n=f[\chi u_{n-1}-\lambda_n\Phi(u_{n-1})] \text{ in } V; \qquad (3.9)$$
$$\frac{\partial u_n}{\partial n}+Bhu_n=0 \text{ on } S.$$

The bounds (3.5) and (3.7) indicate the need for setting $\chi=\chi_0$. The solution (3.8) is written in the form

$$u_0=Cv_0, \qquad (3.10)$$

where v_0 is the orthonormalized eigenfunction (3.8) corresponding to $\chi=\chi_0$. We substitute Eq. (3.10) in (3.9) for $n=1$, obtaining, from the existence condition for the solution of the problem,

$$\lambda_0=\frac{\chi_0 C}{\int\limits_V f\Phi(Cv_0)\,v_0 dV}. \qquad (3.11)$$

Since C is an arbitrary constant, we easily prove that we may require, without loss of generality, that the following conditions are satisfied:

$$\int\limits_V f u_n v_0 dV=0 \quad (n=1,2,\ldots).$$

Then the solutions of the system (3.9) can be represented in the form

$$u_1=Cv_0+\sum_{j=1}^{\infty}\frac{\alpha_j}{\chi_j-\chi_0}v_j \quad \left(\alpha_j=\int\limits_V[\chi_0u_0-\lambda_0\Phi(u_0)]\,v_j dV\right)\ldots,$$

where χ_j and v_j are the eigenvalues and eigenfunctions of the problem. As a result the existence conditions for the solutions of the problem (3.9) lead to the sequence

$$\lambda_{n-1}=\chi_0\int\limits_V fu_{n-1}v_0dV\Big/\int\limits_V f\Phi(u_{n-1})\,v_0dV. \qquad (3.12)$$

$\lambda_n(C)\to\lambda(C)$ as $n\to\infty$.

Thus, the construction of the dependence $\lambda(C)$ reduces in each approximation to the calculation of quadratures in Eqs. (3.11) and (3.12). A different choice of χ can be made if the ratio $u/\Phi(u)$ is unbounded. For example, if there exists a number u_0, such that $\Phi(u_0)=0$, it is necessary to replace χ_0 by χ_j ($j=1, 2,\ldots$) in the equations presented above, i.e., every eigenvalue of the problem (3.6) is a point of bifurcation of the system (3.2), (3.4). In analyzing the dependence $\lambda(C)$, we are able to determine the critical values λ_* and investigate the behavior of a bifurcation of the solutions of the nonlinear problem. The eigenvalues and eigenfunctions of the linear function can be found on the basis of well-known methods [9].

§ 4. A numerical calculation was carried out for $L=0.05$ m, $y_0=0.88$, and $\nu=0.5$. The infinite system was held to 100 unknowns x_n and y_j. In using the method of successive approximations, the first six signs of the unknowns were interchanged after 10-12 iterations. The resulting solution satisfied the boundary conditions to four significant digits. The convergence of the series was improved using the Krylov method [9] for calculating stresses on boundary surfaces.

Figures 1 and 2 depict the distributions of the normal and tangential stresses to the plate. We may see that a nearly homogeneous stressed state occurs only in the central part of the body. The singularity results in a sharp increase in stresses as the corner is approached. The dissipation function behaves analogously in this region. Values of f calculated using Eq. (3.3) are depicted in Fig. 3. In order to determine the temperature the problem (2.1), (2.2) was solved by the method of finite differences with variable pitch for $h_1=40$ m^{-1} and $h_2=5240$ m^{-1}. With these remarks in mind, the mean integral value of f was calculated in an angular grid $0.02L\cdot0.02Ly_0$ using Eq. (2.3). Figure 4 depicts a stationary temperature field calculated to within the factor $r=\omega\mu_2\gamma a_0^2L^2A/k$.

Under these conditions the influence of a fringe effect on temperature is not practically exerted and temperature reaches a maximum at the central point of approach. The eigenvalues and eigenfunctions χ_j and v_j of the linear problem (3.6) were determined by the Bubnov method in the course of studying thermal instability. The approximate solutions are represented in the form

$$v_j^{(s)}=\sum_{n=1}^{s}\sum_{m=1}^{s}b_{nm}\cos\lambda_m x\cos k_n y, \qquad (4.1)$$

where λ_m and k_n are roots of transcendental equations obtained from the boundary conditions (3.2). The function f(x, y) was pointwise approximated using the method of least squares with sums of the form (4.1). As a result, we obtained the following approximations $\chi_0^{(s)}$ of χ_0: $\chi_0^{(1)}=1.0095$, $\chi_0^{(2)}=0.99605$, and $\chi_0^{(3)}=0.99557$, which gives us grounds for setting $\chi_0=0.9956$.

For a broad class of viscoelastic materials [17], $\Phi(u)=\exp(u)$ in Eq. (3.1). The curve for $\lambda(C)$ obtained for this dependence is depicted in Fig. 5. Curves 1 and 2 were obtained using Eqs. (3.7) and (3.11), respectively. Both equations satisfactorily agree in maxima of λ and significantly diverge in the amplitudes of C. The dependence $\lambda(C)$ is a branching equation for the nonlinear problem (3.2), (3.3) and demonstrates that there exist two solutions when $\lambda<\lambda_*$, one solution when $\lambda=\lambda_*$, and no solution when $\lambda>\lambda_*$. When $\lambda>\lambda_*$ the temperature growth is unbounded (thermal explosion).

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